The Fractional Fourier Transform and Wave Propagation

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Abstract

A brief introduction to the fractional Fourier transform and its relation to wave and beam propagation is given. Its relation to phase-space representations (time- or space-frequency representations) and the concept of fractional Fourier domains are discussed. An overview of selected applications is also included.

1 Introduction

Since the ordinary Fourier transform and related techniques find widespread use in physics and engineering, it is natural to expect the fractional Fourier transform (FRT) to find many applications as well. While the FRT has already found many applications in the areas of signal processing and optics following its introduction to these communities in the early nineties [1, 2, 3, 4], applications in other areas remain to be explored. Those interested in learning more about the FRT are referred to [5] or the shorter [6].

The FRT is a generalization of the ordinary Fourier transform with an order (or power) parameter a. The ath order fractional Fourier transform operator is the ath power of the ordinary Fourier transform operator. If we denote the ordinary Fourier transform operator by $\mathcal{F}$, then the ath order fractional Fourier transform operator is denoted by $\mathcal{F}^a$. The zeroth-order fractional Fourier transform operator $\mathcal{F}^0$ is equal to the identity operator $\mathcal{I}$. The first-order fractional Fourier transform operator $\mathcal{F}^1$ is equal to the ordinary Fourier transform operator. Integer values of a correspond to repeated application of the Fourier transform; for instance, $\mathcal{F}^2$ corresponds to the Fourier transform of the Fourier transform. $\mathcal{F}^{-1}$ corresponds to the inverse Fourier transform operator. The a'th order transform of the ath order transform is equal to the $(a'+a)$th order transform: $\mathcal{F}^{a'}\mathcal{F}^a = \mathcal{F}^{a'+a}$, a property referred to as index additivity. The order a may assume any real value, however the operator $\mathcal{F}^a$ is periodic in a with period 4; that is $\mathcal{F}^{a+4j} = \mathcal{F}^a$ where j is any integer. This is because $\mathcal{F}^2$ equals the parity operator $\mathcal{P}$ which maps $f(u)$ to $f(-u)$ and $\mathcal{F}^4$ equals the identity operator. Therefore, the range of a is usually restricted to $(-2, 2]$ or $[0, 4)$. Complex-ordered transforms have also been discussed by some authors; however there remains much to do in this area both in terms of theory and applications.

The same facts can also be thought of in terms of the functions which these operators act on. For instance, the 0th order fractional Fourier transform of the function $f(u)$ is merely the function itself, and the 1st order transform is its ordinary Fourier transform $F(\mu)$, where μ denotes the frequency domain variable. The ath fractional Fourier transform of $f(u)$ is denoted by $f_a(u)$ so that $f_0(u) = f(u)$ and $f_1(\mu) = F(\mu)$.

2 Definition

The most straightforward way of defining the FRT is as an integral transform as follows:

$$f_a(u) = \int_{-\infty}^{\infty} K_a(u,u') f(u')\, du', \qquad K_a(u,u') = \sqrt{1 - i\cot\alpha}\, \exp\left[i\pi(\cot\alpha\, u^2 - 2\csc\alpha\, uu' + \cot\alpha\, u'^2)\right], \tag{1}$$

when $a \neq 2j$ and where $\alpha = a\pi/2$. When $a = 4j$ the transform is defined as $K_a(u,u') = \delta(u-u')$ and when $a = 4j+2$ the transform is defined as $K_a(u,u') = \delta(u+u')$. It can be shown that the above definition indeed corresponds to the operator power of the ordinary Fourier transform [5].

3 Fractional Fourier domains

One of the most important concepts in Fourier analysis is the concept of the Fourier (or frequency) domain. This "domain" is understood to be a space where the Fourier transform representation of the signal lives, with its own interpretation and qualities. This naturally leads one to inquire into the nature of the domain where the fractional Fourier transform representation of a function lives. This is best understood by considerind the phase space spanned by the axes u (usually time or space) and μ (temporal or spatial frequency). This phase space is also referred to as the time-frequency or space-frequency plane in the signal processing literature. The horizontal axis u is simply the time or space domain, where the original function lives. The vertical axis μ is simply the frequency (or Fourier) domain where the ordinary Fourier transform of the function lives. Oblique axes making angle α constitute domains where the ath order fractional Fourier transform lives, where a and α are related through $\alpha = a\pi/2$.

For those familiar with phase spaces from a mechanics—rather than signal analysis—perspective, we note that the correspondence between spatial frequency and momentum allows one to construct a correspondence between the familiar mechanical phase space of a single degree of freedom (defined by the space and momentum axes), and the phase space of signal analysis (defined by the space and spatial frequency axes).

Referring to axes making angle $\alpha = a\pi/2$ with the u axis as the "ath fractional Fourier domain" is supported by several of the properties of the fractional Fourier transform [5]. However, the most substantial justification is based on the fact that *fractional Fourier transformation corresponds to rotation in phase space*. This can be formulated in many ways, the most straightforward being to consider a phase-space distribution (or time/space-frequency representation) of the function $f(u)$, such as the Wigner distribution $W_f(u, \mu)$, which is defined as

$$W_f(u, \mu) = \int f(u + u'/2) f^*(u - u'/2) e^{-i2\pi\mu u'}\, du'. \tag{2}$$

The many properties of the Wigner distribution support its interpretation as a function giving the distribution of signal energy in phase space. Now, it is possible to show that the Wigner distribution $W_{f_a}(u, \mu)$ of $f_a(u)$ is a clockwise rotated version of the Wigner distribution $W_f(u, \mu)$ of $f(u)$:

$$W_{f_a}(u, \mu) = W_f(u \cos\alpha - \mu \sin\alpha, u \sin\alpha + \mu \cos\alpha). \tag{3}$$

That is, the act of fractional Fourier transformation on the original function, corresponds to rotation of the Wigner distribution.

4 Applications in wave propagation, diffraction, and optics

It has been shown that there exists a fractional Fourier transform relation between the (appropriately scaled) complex amplitude distributions on two spherical reference surfaces with given radii and separation. This result provides an alternative statement of the law of propagation. While originally formulated in an optical context, the same result equally applies to electromagnetic and acoustic waves satisfying the linear wave equation. One of the central results of diffraction theory is that the far-field diffraction pattern is the Fourier transform of the diffracting object. It is possible to generalize this result by showing that the field patterns at closer distances are the fractional Fourier transforms of the diffracting object. As the wavefield propagates, its distribution evolves through fractional transforms of increasing orders. In other words, the process of diffraction can be modeled as that of continual fractional Fourier transformation [7, 8, 9, 10, 11].

More generally, in an optical system involving many lenses separated by arbitrary distances, it is possible to show that the amplitude distribution is continuously fractional Fourier transformed as it propagates through the system. The order $a(z)$ of the fractional transform observed at the distance z along the optical axis is a continuous monotonic increasing function. Just as the case with free-space propagation, the distribution of light evolves through fractional transforms of increasing orders. Wherever the order of the transform $a(z)$ is equal to $4j + 1$ for any integer j, we observe the Fourier transform of the input. Wherever the order is equal to $4j + 2$, we observe an inverted image, etc. [7].

Propagation in graded-index media, and Gaussian beam propagation can also be studied in terms of the fractional Fourier transform [1, 8, 12, 13].

The fractional Fourier transform can be optically realized in a similar manner as the common Fourier transform. The fact that the fractional Fourier transform can be realized optically means that the many applications of the transform in signal processing can also be carried over to optical signal processing.

Other optical applications include spherical mirror resonators (lasers) [8], optical systems and lens design [14], quantum optics [15, 16], phase retrieval [16, 17], statistical optics [18], and beam shaping [19].

5 Other applications

The transform has found widespread use in signal and image processing, in areas ranging from time/space-variant filtering, perspective projections, phase retrieval, image restoration, pattern recognition, tomography, data compression, encryption, watermarking, and so forth (see [5] for references). Concepts such as "fractional convolution" and "fractional correlation" have been studied. One of the most striking applications is that of filtering in fractional Fourier domains. In traditional filtering, one takes the Fourier transform of a signal, multiplies it with a Fourier-domain transfer function, and inverse transforms the result. Here, we take the fractional Fourier transform, apply a filter function in the fractional Fourier domain, and inverse transform to the original domain. It has been shown that considerable improvement in performance is possible by exploiting the additional degree of freedom coming from the order parameter a. This improvement comes at no additional cost since computing the fractional Fourier transform is not more expensive than computing the ordinary Fourier transform [5].

The fractional Fourier transform is intimately related to the harmonic oscillator in both its classical and quantum-mechanical forms. The kernel $K_a(u, u')$ given in equation 1 is precisely the Green's function (time-evolution operator kernel) of the quantum-mechanical harmonic oscillator differential equation. In other words, the time evolution of the wave function of a harmonic oscillator corresponds to continual fractional Fourier transformation. In classical mechanics, the relationship can be most easily seen by noting that—with properly normalized coordinates—the phase space point describing harmonic oscillation follows circular trajectories; that is, it rotates in phase space. Therefore, one can expect the fractional Fourier transform to play an important role in the study of vibrating systems, an application area which has so far not received attention.

Another potential application area is the solution of differential equations. Namias and McBride and Kerr [20, 21, 22] have shown how the fractional Fourier transform can be used to solve certain differential equations. Constant coefficient (shift-invariant) equations can be solved with the ordinary Fourier or Laplace transforms. It has been shown that certain kinds of second-order differential equations with non-constant coefficients can be solved by exploiting the additional degree of freedom associated with the order parameter a. One proceeds by taking the fractional Fourier transform of the equation and then choosing a such that the second-order term disappears, leaving a first-order equation whose exact solution can always be written. Then, an inverse transform (of order $-a$) provides the solution of the original equation.

We believe that the fractional Fourier transform is of potential usefulness in every area in which the ordinary Fourier transform is used. The typical pattern of discovery of a new application is to concentrate on an application where the ordinary Fourier transform is used and ask if any improvement or generalization might be possible by using the fractional Fourier transform instead. The additional order parameter often allows better performance or greater generality because it provides an additional degree of freedom over which to optimize.

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